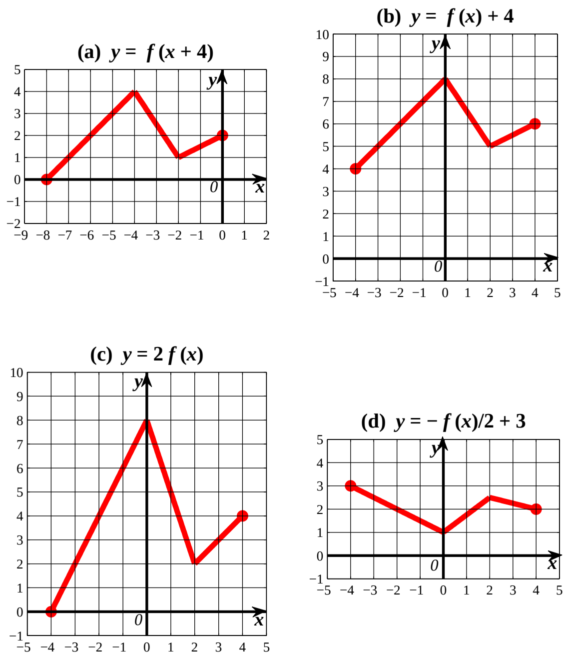
HW1 SOLUTIONS

MAT 1320D Winter 2009

Problem 1.



Problem 2. $f(x) = \sqrt{2x+3},\ g(x) = x^2 + 1$

(1) $(f \circ g)(x) = f(g(x)) = f(x^2+1)$

$= \sqrt{2(x^2+1)+3} = \sqrt{2x^2+5}$

$\therefore (f \circ g)(x) = \sqrt{2x^2+5}$

▶ Domain: $\mathbb{R}$ or $(-\infty, \infty)$

(2) $(g \circ f)(x) = g(f(x)) = g(\sqrt{2x+3})$

$= 2x + 3 + 1 = 2x + 4$

$\therefore (g \circ f)(x) = 2x + 4$

▶ Domain: $[-3/2, \infty)$ or $\{x |\ x \geq -3/2\}$

(3) $(f \circ f)(x) = f(f(x)) = f(\sqrt{2x+3})$

$= \sqrt{2\sqrt{2x+3}+3}$

$\therefore (f \circ f)(x) = \sqrt{2\sqrt{2x+3}+3}$

▶ Domain: $[-3/2, \infty)$ or $\{x |\ x \geq -3/2\}$

(4) $(g \circ g)(x) = g(g(x)) = g(x^2+1)$

$= (x^2+1)^2 + 1 = x^4 + 2x^2 + 2$

$\therefore (g \circ g)(x) = x^4 + 2x^2 + 2$

▶ Domain: $\mathbb{R}$ or $(-\infty, \infty)$

Problem 3.

(a) $f(g(1)) = f(6) = 5$

(b) $g(f(1)) = g(3) = 2$

(c) $f(f(1)) = f(3) = 4$

(d) $g(g(1)) = g(6) = 3$

(e) $(g \circ f)(3) = g(f(3)) = g(4) = 1$

(f) $(f \circ g)(6) = f(g(6)) = f(3) = 4$

Problem 4.

(t in hour)

$t = 0\ :\ m(0) = 2$

$t = 15\ :\ m(15) = \frac{1}{2} \cdot 2$

$t = 30\ :\ m(30) = \frac{1}{2} \cdot \frac{1}{2} \cdot 2 = \frac{1}{2^2} \cdot 2$

$t = 45\ :\ m(30) = \frac{1}{2} \cdot \frac{1}{2^2} \cdot 2 = \frac{1}{2^3} \cdot 2$

$\vdots$

$\therefore m(t) = \frac{1}{2^{t/15}} \cdot 2 = 2 \cdot 2^{-t/15}$

(a) $t = 60$ hours

$m(60) = 2 \cdot 2^{-60/15} = 2 \cdot 2^{-4} = 2^{-3} = \frac{1}{8}$.

$\therefore m(60) = 1/8$ g.

(b) In t hours, there will be $t/15$ half-life periods. The initial mass is 2 g, so the mass at time t is

$$m(t) = 2 \cdot 2^{-t/15} \left(= 2^{1-t/15}\right).$$

(c) 4 days $\Rightarrow$ 4· 24 = 96 hours

$t = 96:\ m(96) = 2 \cdot 2^{-96/15} \approx 0.024$ g

(d) $m(t) = 2 \cdot 2^{-t/15} = 0.01$

$\Rightarrow 2^{-t/15} = 0.005$

$\Rightarrow \log_2 2^{-t/15} = \log_2 0.005$

$\Rightarrow -t/15 = \log_2 0.005$

$\Rightarrow t = -15 \log_2 0.005 \approx 114.7$

$\therefore t \approx 114.7$ hours.

Problem 5.

▶ Solve for x in terms of y:

$y = \frac{1+e^x}{1-e^x} \Rightarrow y(1-e^x) = 1 + e^x$

$\Rightarrow y - 1 = ye^x + e^x$

$\Rightarrow y - 1 = e^x(y+1)$

$\Rightarrow e^x = \frac{y-1}{y+1}$

$\Rightarrow x = \ln \frac{y-1}{y+1}$

▶ Interchange x and y: $y = \ln \frac{x-1}{x+1}$

$\therefore f^{-1}(x) = \ln \frac{x-1}{x+1}$

Problem 6.

(a) $Q(t) = Q_0\left(1 - e^{-t/a}\right)$

$$\Rightarrow \frac{Q}{Q_0} = 1 - e^{-t/a}$$

$$\Rightarrow e^{-t/a} = 1 - \frac{Q}{Q_0}$$

$$\Rightarrow -t/a = \ln\left(1 - \frac{Q}{Q_0}\right)$$

$$\Rightarrow t = -a\ln\left(1 - \frac{Q}{Q_0}\right)$$

This gives us the time necessary to obtain a given charge Q.

(b) $Q = 0.9Q_0$ and $a = 2$:

$$t = -2\ln\left(1 - \frac{0.9Q_0}{Q_0}\right) = -2\ln(1 - 0.9)$$

$\therefore\ t = -2\ln 0.1 \approx 4.6$ seconds.